Stochastic Properties of Conflict Frequency at Multiple Connected Air Route Intersections

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Previous results regarding the stochastic properties of the frequency of conflict situations at a single intersection of two air routes have been extended to the case of multiple connected intersections. Such a situation occurs when more than one aircraft stream intersects with another aircraft stream. Examples include multiple discrete velocity groups of vehicles on routes, intersections in the horizontal plane involving more than two routes, and routes transitioning through altitude strata or flight levels. It is determined that the conflict frequency at multiple intersections connected with a common vehicle stream are correlated, and the expression for the covariance is derived. With these results, not only the mean, but the variance of conflict frequency for an ATC sector may be predicted from traffic-flow and route-geometry information, both of which are important parameters for sector capacity analyses.

Introduction

In this investigation, we are concerned with the stochastic properties of the frequency of conflicts occurring at intersections involving numerous air routes. Analyzing and predicting conflict frequency is important in evaluating air traffic controller workload as well as in the design of automation features such as automatic conflict alert algorithms.

It has been previously shown[1] that the stochastic properties of the vehicle conflict† frequency at an intersection of two air routes can be investigated by expressing the conflict frequency as a random sum of correlated random variables. Using this approach, exact expressions for the mean, variance, and the approximate distribution for the number of conflicts in a specified time period were derived. It was found that, under the assumption of Poisson traffic arrivals, the variance of conflict frequency was always larger than the mean, and that the distribution was approximately compound Poisson. (The Poisson traffic assumption was justified in Ref. 2.) The expression for the mean conflict frequencies gave results identical to those of previous investigators,[2,3] but the results concerning the variance and distribution represented new findings.

However, the analysis of an entire ATC sector and the workload of the sector controller is the primary motivation for this investigation (see, for example, Refs. 4 and 5). Therefore, the treatment of more than one intersection and different velocities on the routes is an order.

In this paper the stochastic properties of the conflict frequency for a set of connected intersections will be developed. First, the case of multiple speed groups rather than a single velocity on a route will be addressed. Then the case of multiple intersections joined by a common route will be shown to be analogous to the preceding (multiple speed) situation. With these results, the conflict-frequency statistics (i.e., mean and variance) for an entire sector may be obtained.

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†Here a conflict is defined as an aircraft separation situation that would be perceived by the controller to require action to avoid violating the separating minima. In the enroute airspace, this separation is approximately 9 n. mi., while the horizontal separate minimum is actually 5 n. mi.

Summary of Previous Findings

Consider an intersection of two air routes as shown in Fig. 1. Aircraft are traveling on routes 1 and 2 at velocities V_1 and V_2, respectively, and the average aircraft flow rates through the intersection are λ_1 and λ_2. If the number of aircraft through the intersection on either route in a specified time t is assumed to be a Poisson distributed random variable, its mean and variance is equal to λt.

Now consider the ith aircraft through the intersection on route 1. It has been shown[1] that this aircraft will, at some time, be perceived in conflict with any aircraft on the segment D_i of route 2. Furthermore, the length of segment D_i has been shown to be

$$D=\frac{2A\sqrt{V_1^2+V_2^2-2V_1V_2\cos\alpha}}{V_1\sin\alpha} \tag{1}$$

for intersections with the geometry shown, where A is the effective separation distance which defines a perceived conflict (e.g., 9 n. mi.). Since the number of aircraft in segment D_i is a random variable, m_i, and the number of aircraft through the intersection on route 1 in time t is also a random variable, n_1, the number of perceived conflicts N in time t is

$$N=\sum_{i=1}^{n_1} m_i \tag{2}$$

To show that the number of conflicts with the ith aircraft, m_i, is not independent of those with the jth aircraft, m_j, consider Fig. 2. If the time between the ith and jth, is such that their conflict route segments D_i and D_j overlap as shown, any aircraft in the overlapping segment route 2 would at some time be in conflict with both aircraft i and j (i.e., be included in both m_i and m_j).

Finally, with the conflict frequency as given by Eq. (2), it was easily shown that

$$E(N)=E(n_1)E(m) \tag{3}$$

with

$$E(n_1)=\lambda_1 t$$

$$E(m)=\lambda_2 D/V_2=\lambda_2 t_d$$

and

$$\mathrm{Var}(n)=E_{n_1}[E(N^2 \mid n_1)]-E^2(N)$$

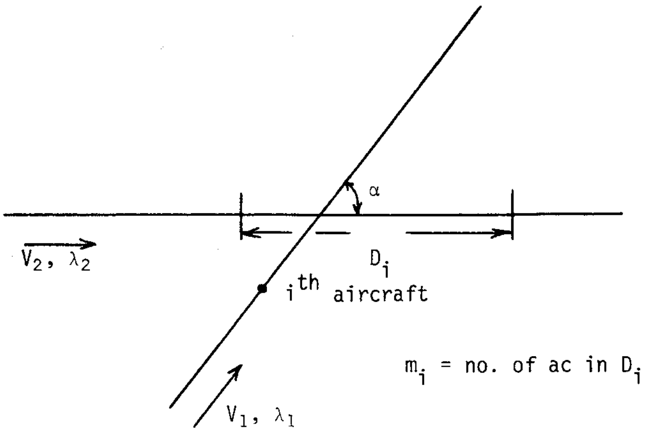

Fig. 1 Intersection geometry.

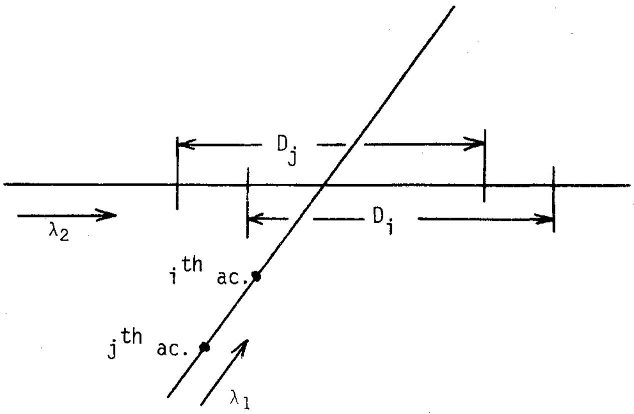

Fig. 2 Intersection with overlapping track intersection distances.

or

$$\mathrm{Var}(N) = E_{n_1}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_i m_j)\right] - E^2(N) \tag{4}$$

Finally, using the concept of the "double-counted" aircraft discussed earlier, it was found that

$$\mathrm{Var}(N) = E(N)\,[1+E(m)] + 2\sum_{k=1}^{\infty} C_K(t)\,C_K(t_d)\left(\frac{\lambda_2}{\lambda_1}\right) \tag{5}$$

where

$$C_K(t) = \lambda_1 t - K + \sum_{j=0}^{k-1} (K\text{-}j)\,\frac{(\lambda_1 t)^j e^{-\lambda_1 t}}{j!}$$

Generalization to Multiple Speed Groups

Now, consider the vehicles traveling on route 2, in Fig. 1, to consist of two *independent* groups with mean flow rates $\lambda_{2,a}$ and $\lambda_{2,b}$, and velocities $V_{2,a}$ and $V_{2,b}$. Assume, for the moment, that only a single velocity group is on route 1.

Now, as the ith aircraft on route 1 travels through the intersection, its track intersection distances with respect to aircraft group a would be D_a, and the distance with respect to group b would be D_b, where D_a and D_b can each be found from Eq. (1). Finally, denote the number of group a aircraft on route 2 in D_a as $m_{i,a}$, and the number of group b aircraft in D_b as $m_{i,b}$. Now, the total number of conflicts can be expressed as

$$N = \sum_{i=1}^{n_1} (m_{i,a} + m_{i,b}) = \sum_{i=1}^{n_1} m_{i,a} + \sum_{i=1}^{n_1} m_{i,b} = N_a + N_b \tag{6}$$

where N_a is the number of conflicts involving group a aircraft, N_b, is that involving group b aircraft, and

$$N_a = \sum_{i=1}^{n_1} m_{i,a}$$

$$N_b = \sum_{i=1}^{n_1} m_{i,b}$$

Now, the expected value of N is

$$E(N) = E(N_a) + E(N_b) = E(n_1)\,[E(m_a) + E(m_b)] \tag{7}$$

where

$$E(n_1) = \lambda_1 t$$

$$E(m_a) = \lambda_{2,a}\frac{D_a}{V_{2,a}} = \lambda_{2,a} t_{d,a}$$

$$E(m_b) = \lambda_{2,b}\frac{D_b}{V_{2,b}} = \lambda_{2,b} t_{d,b}$$

Clearly, if $t_{d,a} = t_{d,b} = t_d$

$$E(N) = \lambda_1 t(\lambda_{2,a} + \lambda_{2,b})t_d = \lambda_2 t_d$$

which could be obtained directly from Eq. (3). So we see that this generalization is consistent with the previous results.

Now, the variance of N as given by Eq. (6) may be expressed as

$$\mathrm{Var}(N) = \mathrm{Var}(N_a) + \mathrm{Var}(N_b) + 2\mathrm{Cov}(N_a N_b) \tag{8}$$

and, as we shall find, $\mathrm{Cov}(N_a N_b) \neq 0$.

Using Eq. (4), and realizing that

$$m_i = m_{i,a} + m_{i,b}$$

we can express Var (N) as

$$\mathrm{Var}(N) = E_{n_1}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,a}m_{j,a}) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,b}m_{j,b}) + 2\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,a}m_{j,b})\right] - E^2(N) \tag{9}$$

and from Eq. (7),

$$E^2(N) = E^2(n_1)\,[E^2(m_a) + E^2(m_b) + 2E(m_a)E(m_b)]$$

Since, from Eq. (4),

$$\mathrm{Var}(N_a) = E_{n_1}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,a}M_{j,a})\right] - E^2(N_a)$$

and

$$\mathrm{Var}(N_b) = E_{n_1}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,b}m_{j,b})\right] - E^2(N_b)$$

and comparing Eqs. (8) and (9), we see that

$$\mathrm{Cov}(N_a N_b) = E_{n_1}\left[\sum_{i=1}^{n_1}\sum_{j=1}^{n_1} E(m_{i,a}m_{j,b})\right] - E^2(n_1)E(m_a)E(m_b) \tag{10}$$

or, as well known,

$$\mathrm{Cov}(N_a N_b) = E(N_a N_b) - E(N_a)E(N_b)$$

Now, to evaluate $E(m_{i,a}m_{j,b})$ consider Fig. 2. If $m_{i,a}$ is the number of aircraft of group a in the intersection distance D_i, and $m_{j,b}$ is the number of group b aircraft in the segment D_j, $m_{i,a}$ and $m_{j,b}$ are independent, even if the intersection distances overlap. That is, since we are considering two independent aircraft streams, a and b, no aircraft are "double-counted" in the overlapping segment. [However, this "double-counting" still occurs when considering $E(m_{i,a}m_{j,a})$ or $E(m_{i,b}m_{j,b})$, but these factors are already included in Var (N_a) and Var (N_b).] Therefore,

$$E(m_{i,a}m_{j,b}) = E(m_{i,a})E(m_{j,b}) = E(m_a)E(m_b)$$

Incorporating this into Eq. (10), we find that

$$\mathrm{Cov}(N_aN_b) = E_{n_l}[n_l^2E(m_a)E(m_b)] - E^2(n_l)E(m_a)E(m_b)$$

or

$$\mathrm{Cov}(N_aN_b) = \mathrm{Var}(n_l)E(m_a)E(m_b) \quad (11)$$

Finally, from Eq. (8), we have

$$\mathrm{Var}(N) = \mathrm{Var}(N_a) + \mathrm{Var}(N_b) + 2\mathrm{Var}(n_l)E(m_a)E(m_b) \quad (12)$$

or, for Poisson traffic on route 1

$$\mathrm{Var}(N) = \mathrm{Var}(N_a) + \mathrm{Var}(N_b) + 2(\lambda_l t)(\lambda_{2,a}t_{d,a})(\lambda_{2,b}t_{d,b})$$

Compatibility of this generalization with Eq. (5) for the single-speed-group case may be readily shown by letting $t_{d,a} = t_{d,b} = t_d$, and by noting Eq. (7), $E(m) = E(m_a) + E_{m_b}$, and $\lambda_2 = \lambda_{2,a} + \lambda_{2,b}$.

The results expressed in Eq. (12) indicate that the variance of the number of conflicts involving two independent aircraft streams on one route may be obtained from the variance of conflict frequency for each stream treated separately, plus the addition of the covariance term given in Eq. (11). Note that in this latter expression, the term Var (n_l) is the variance of the traffic on the route intersected by *both aircraft streams a* and *b*. That is, the Var (n_l) refers to the "connecting" route. Clearly, if there is no "connecting" route, N_a and N_b would be independent.

These results [i.e., Eqs. (7) and (12)], may be extended directly to multiple aircraft streams on both routes by dealing with each stream independently and summing the results. Then, in calculating the variance of total conflict frequency, a covariance [Eq. (11)] term for each pair of streams intersecting a connecting stream would be added to the sum of the individual variances. For example, if two streams, a and b, were on one route, and two other streams, c and d, were on the other route, and if we define $N_{a,c}$ as the number of conflicts between streams a and c, the mean and variance of total conflict frequency would be

$$E(N) = E(N_{a,c}) + E(N_{a,d}) + E(N_{b,c}) + E(N_{b,d})$$

and

$$\mathrm{Var}(N) = \mathrm{Var}(N_{a,c}) + \mathrm{Var}(N_{a,d}) + \mathrm{Var}(N_{b,c}) + \mathrm{Var}(N_{b,d}) + 2[\mathrm{Cov}(N_{a,c}N_{a,d}) + \mathrm{Cov}(N_{b,c}N_{b,d}) + \mathrm{Cov}(N_{a,c}N_{b,c}) + \mathrm{Cov}(N_{a,d}N_{b,d})]$$

Now, each of the terms in the preceding expression may be evaluated by Eqs. (3, 5, or 11).

Generalization to Multiple Connected Intersections

Consider the multiple intersection gometries shown in Fig. 3. If, for example, we consider the ith aircraft traveling on route 1 of Fig. 3a, it would be in conflict with any aircraft on route 2 in a segment of length D_2. Also, it would conflict with any aircraft on route 3 in a segment of length D_3. Now, since the traffic streams on routes 2 and 3 are *independent,* we have a situation exactly analogous to the multiple-speed-group case presented in the previous section. That is, the multiple-speed-group case is a special case of a multiple connected intersection. The only difference results from the fact that all three streams intersect in this case, while in the simplest multiple-speed-group case, there were only two sets of intersecting streams. As a result of the analogy, the conflict frequency and its mean and variance may be written by inspection as

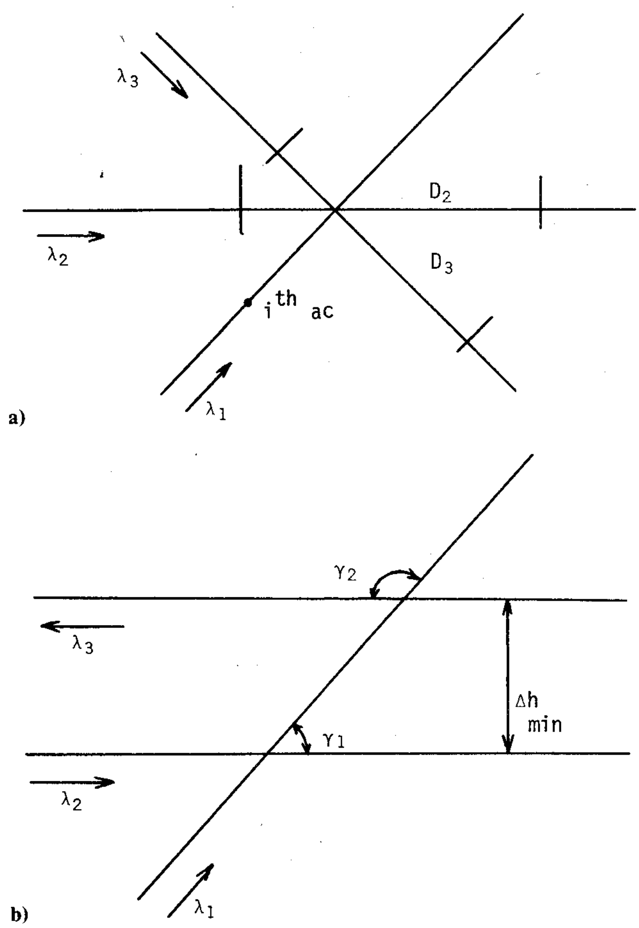

Fig. 3 Multiple intersection geometries.

$$N = N_{1,2} + N_{1,3} + N_{2,3}$$

$$E(N) = E(N_{1,2}) + E(N_{1,3}) + E(N_{2,3})$$

$$\mathrm{Var}(N) = \mathrm{Var}(N_{1,2}) + \mathrm{Var}(N_{1,3}) + \mathrm{Var}(N_{2,3}) + 2[\mathrm{Cov}(N_{1,2}N_{1,3}) + \mathrm{Cov}(N_{1,2}N_{2,3}) + \mathrm{Cov}(N_{1,3}N_{2,3})]$$

Again, each of these terms may be evaluted using Eqs. (3, 5, and 11).

With regard to the multiple intersection geometry of Fig. 3b, it should be noted that this geometry not only occurs in the horizontal plane, but the vertical plane as well. The latter occurs, of course, when route 1 (in Fig. 3b) is a transition route for aircraft climbing (or descending) through the different altitude strata or flight levels (as represented by routes 2 and 3). In the transitioning case, however, it was shown[1] that instead of Eq. (1), the track intersection distance on each of the level routes was

$$D_T = 2[A + abs(V_l\cos\gamma - V_{\text{level}})\Delta h_{\text{min.}}/V_l\sin\gamma]$$

where $\Delta h_{\min.}$ is the altitude difference between flight levels, and γ is the climb or flight-path angle of transition.

Summary and Conclusion

In this paper, we have generalized the results for the mean and variance of the conflict frequency at a single intersection to sets of multiple intersections connected by a common aircraft stream. We have found that when a connecting stream exists, the conflict frequency at the various intersections are correlated, and the expression for the covariance was developed in terms of known parameters of the traffic and intersection geometry. Finally, since an ATC sector includes multiple connected as well as independent intersections of the types considered, the mean and variance of the conflict frequency for a sector may now be predicted from knowledge of the traffic flow and route geometry.

To indicate the potential magnitude of these variables, consider three routes intersecting at a particular altitude over a VOR, for example, with the geometry as shown in Fig. 3a. Assume the following: $V_1 = V_2 = V_3 = 480$ knots, $\alpha_{1,2} = \alpha_{2,3} = 60$ deg, $\alpha_{1,3} = 120$ deg, $\lambda_1 = \lambda_3 = 12$ aircraft/hour, $\lambda_2 = 9$ aircraft/hour, $A = 9$ n. mi. With these values the track intersection distances, found from Eq. (1), are $D_{2,1} = D_{2,3} = 20.78$ n.mi., and $D_{3,1} = 36.00$ n.mi., where $D_{i,j}$ is the intersection distance on route i relative to flow on route j. In this case, the mean conflict frequency in one hour (i.e., $t = 1$) is

$$E(N) = \lambda_1 \lambda_2 \frac{D_{2,1}}{V_2} t + \lambda_3 \lambda_2 \frac{D_{2,3}}{V_2} t + \lambda_1 \lambda_3 \frac{D_{3,1}}{V_3} t$$

$$= 4.68 + 4.68 + 10.80 = 20.16 \text{ conflicts/hour}$$

(Note here, if the conflicts were assumed Poisson, this would also be the estimate of the variance.)

To evaluate the variance terms [Eq. (5)] we must determine values for $C^j_K(t_{d_{i,j}})$, where $t_{d_{i,j}} = D_{i,j}/V_i$. We have, for example,

$$C^1_{K=1}(t_{d_{2,1}}) = \lambda_1 t_{d_{2,1}} - 1 + \exp(-\lambda_1 t_{d_{2,1}}) = 0.114$$

$$C^1_{K=2}(t_{d_{2,1}}) = \lambda_1 t_{d_{2,1}} - 2 + \exp(-\lambda_1 t_{d_{2,1}}) = 0.018$$

$$C^1_{K=3}(t_{d_{2,1}}) = \lambda_1 t_{d_{2,1}} - 3 + \exp(-\lambda_1 t_{d_{2,1}}) [3 + 2(\lambda_1 t_{d_{2,1}}) + (\lambda_1 t_{d_{2,1}})^2/2] = 0.002$$

Also

$$C^j_{K=1}(t) = \lambda_1 t - 1 + \exp(-\lambda_1 t) = 11.00 \qquad (j = 1,2,3)$$

$$C^j_{K=2}(t) = \lambda_1 t - 2 + \exp(-\lambda_1 t)[2 + \lambda_1 t] = 10.00 \quad (j = 1,2,3)$$

$$C^j_{K=3}(t) = \lambda_1 t - 3 + \exp(-\lambda_1 t)[3 + 2(\lambda_1 t) + (\lambda_1 t)^2/2] = 9.00 \qquad (j = 1,2,3)$$

Evaluating the remaining terms yields

$$C^1_{K=1}(t_{d_{3,1}}) = 0.307 \qquad C^3_{K=1}(t_{d_{2,3}}) = 0.114$$

$$C^1_{K=2}(t_{d_{3,1}}) = 0.079 \qquad C^3_{K=2}(t_{d_{2,3}}) = 0.018$$

$$C^1_{K=3}(t_{d_{3,1}}) = 0.016 \qquad C^3_{K=3}(t_{d_{2,3}}) = 0.002$$

Therefore, we have

$$\text{Var}(N_{1,2}) = 4.68(1 + .39) + 2(9/12)[11(.114) + 10(.018) + 9(.002)] = 8.69 (\text{conflicts/hr})^2$$

$$\text{Var}(N_{2,3}) = \text{Var}(N_{1,2})$$

$$\text{Var}(N_{1,3}) = 10.80(1 + .90) + 2(12/12)[11(.307) + 10(.079) + 9(.016)] = 29.14 (\text{conflicts/hr})^2$$

Finally, from Eq. (11) we have

$$\text{Cov}(N_{1,2} N_{1,3}) = (\lambda_1 t)(\lambda_2 t_{d_{2,1}})(\lambda_3 t_{d_{3,1}}) = 4.2 (\text{conflicts/hr})^2$$

$$\text{Cov}(N_{1,2} N_{2,3}) = (\lambda_2 t)(\lambda_1 t_{d_{1,2}}) = 2.43 (\text{aircraft/hr})^2$$

$$\text{Cov}(N_{1,3} N_{2,3}) = (\lambda_3 t)(\lambda_1 t_{d_{1,3}})(\lambda_2 t_{d_{2,3}}) = 4.21 (\text{aircraft/hr})^2$$

which yields

$$\text{Var}(N) = 68.22 (\text{conflicts/h})^2$$

over three times the value predict for Poisson conflicts!

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